Interactive Formal Verification
8: Operational Semantics

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Overview

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- The operational semantics of programming languages can be given *inductively*.
    - Type checking
    - Expression evaluation
    - Command execution, including concurrency

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  - Expression evaluation
  - Command execution, including concurrency
- Properties of the semantics are frequently proved by induction.
- Running example: an abstract language with WHILE

Language Syntax

```
typedecl loc -- "an unspecified type of locations"

type_synonym val   = nat -- "values"
type_synonym state = "loc => val"
type_synonym aexp  = "state => val"
type_synonym bexp  = "state => bool"   -- "functions on states"

datatype
  com = SKIP
      | Assign loc aexp         ("_ :== _ " 60)
      | Semi   com com          ("_; _"  [60, 60] 10)
      | Cond   bexp com com     ("IF _ THEN _ ELSE _"  60)
      | While  bexp com         ("WHILE _ DO _"  60)
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Arithmetic & boolean expressions are *functions* over the state

A “Big-Step” Semantics

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$$\frac{\langle c_0, s \rangle \to s'' \qquad \langle c_1, s'' \rangle \to s'}{\langle c_0; c_1, s \rangle \to s'}$$

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$$\frac{b\,s \qquad \langle c_0, s\rangle \to s'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, s\rangle \to s'} \qquad \frac{\neg\, b\,s \qquad \langle c_1, s\rangle \to s'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, s\rangle \to s'}$$

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$$\frac{\neg\, b\,s}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, s\rangle \to s} \qquad \frac{b\,s \quad \langle c, s\rangle \to s'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, s''\rangle \to s'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, s\rangle \to s'}$$

Formalised Language Semantics

```
Com.thy

text {* The big-step execution relation @{text evalc} is defined inductively *}

inductive
  evalc :: "[com,state,state] ⇒ bool" ("⟨_,_⟩/ ⇝ _" [0,0,60] 60)
where
  Skip:    "⟨SKIP,s⟩ ⇝ s"
| Assign:  "⟨x :== a,s⟩ ⇝ s(x := a s)"

| Semi:    "⟨c0,s⟩ ⇝ s'' ⟹ ⟨c1,s''⟩ ⇝ s' ⟹ ⟨c0; c1, s⟩ ⇝ s'"

| IfTrue:  "b s ⟹ ⟨c0,s⟩ ⇝ s' ⟹ ⟨IF b THEN c0 ELSE c1, s⟩ ⇝ s'"
| IfFalse: "¬b s ⟹ ⟨c1,s⟩ ⇝ s' ⟹ ⟨IF b THEN c0 ELSE c1, s⟩ ⇝ s'"

| WhileFalse: "¬b s ⟹ ⟨WHILE b DO c,s⟩ ⇝ s"
| WhileTrue:  "b s ⟹ ⟨c,s⟩ ⇝ s'' ⟹ ⟨WHILE b DO c, s''⟩ ⇝ s'
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lemmas evalc.intros [intro] -- "use those rules in automatic proofs"

-u-:---  Com.thy        17% L24     (Isar Utoks Abbrev; Scripting )-----------------
Wrote /Users/lp15/Dropbox/ACS/8 - Operational Semantics/Com.thy
```

Formalised Language Semantics

an inductive *predicate* with special syntax

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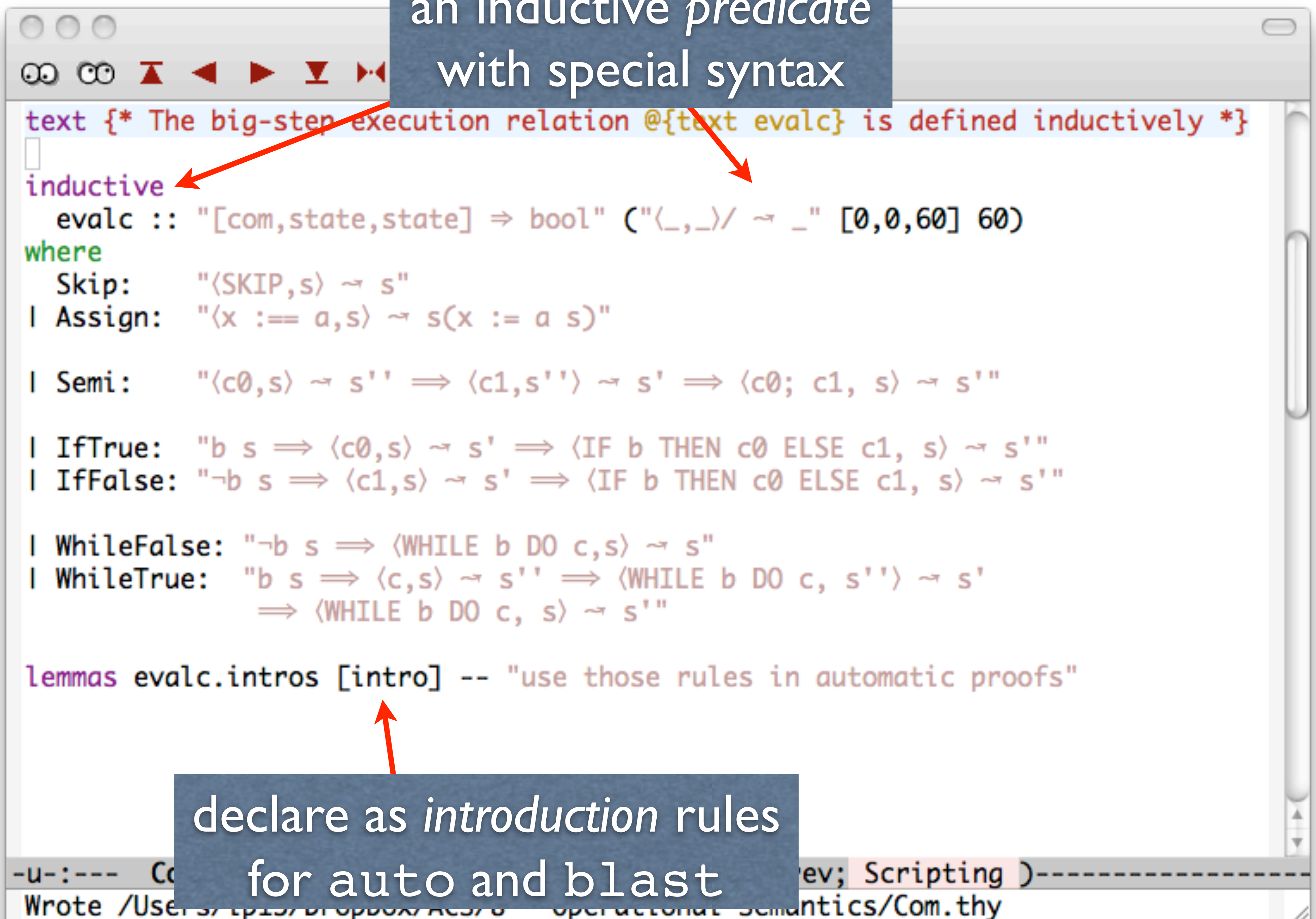
Formalised Language Semantics



an inductive *predicate* with special syntax

declare as *introduction* rules for `auto` and `blast`

Rule Inversion

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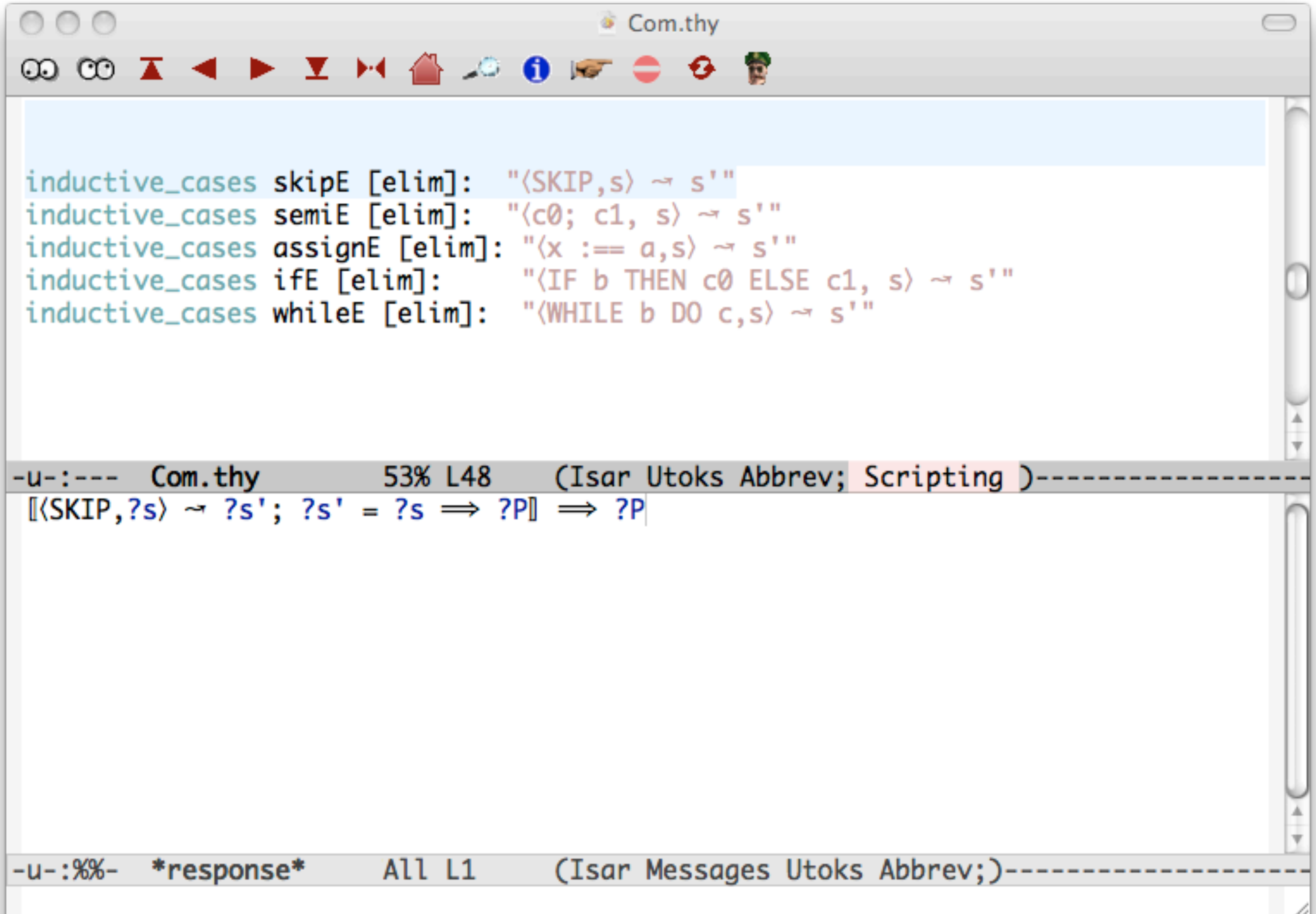
  - b and $\langle c_0, s \rangle \rightarrow s'$, or...
  - $\neg b$ and $\langle c_1, s \rangle \rightarrow s'$
- This sort of case analysis is easy in Isabelle.

Rule Inversion in Isabelle



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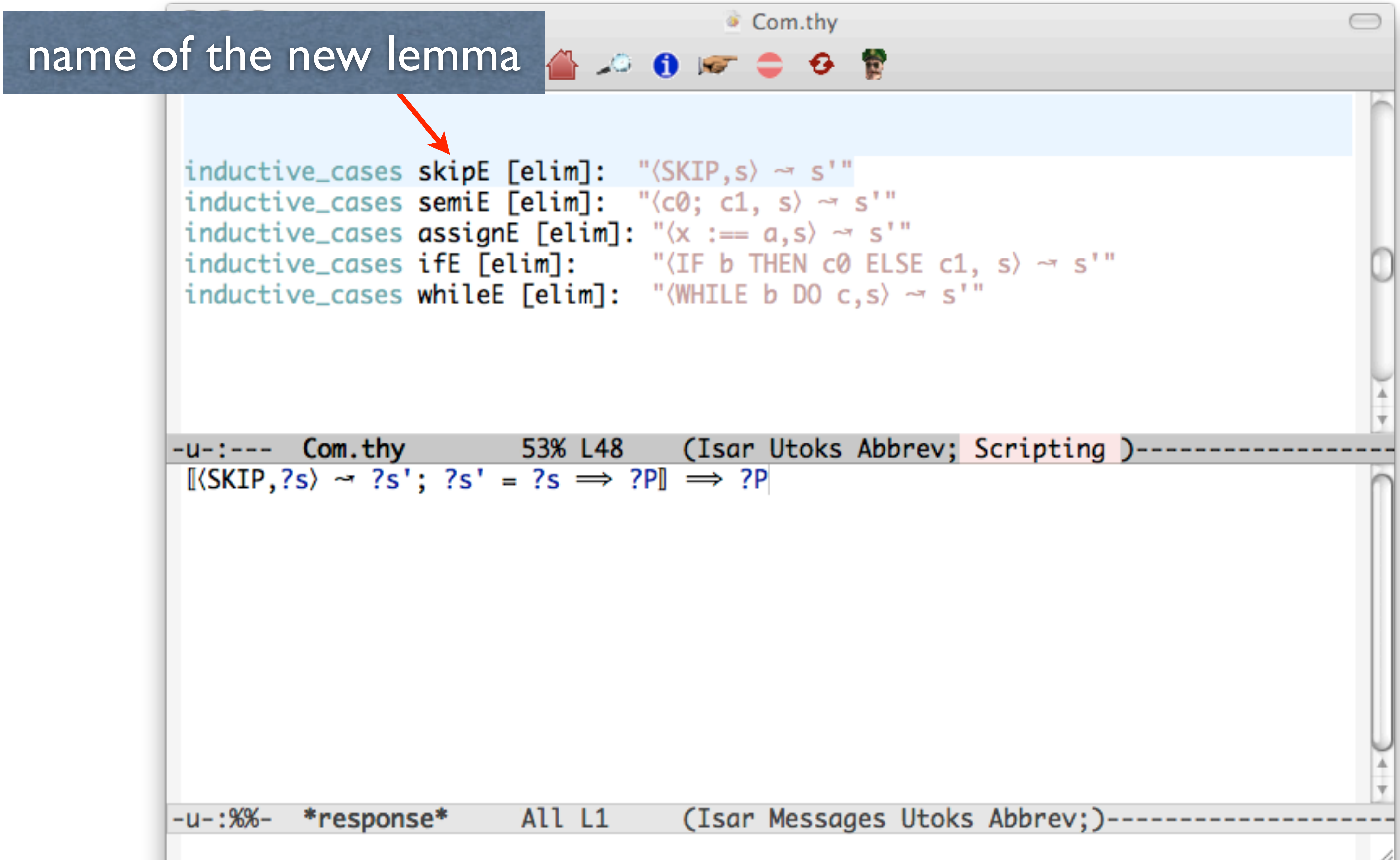

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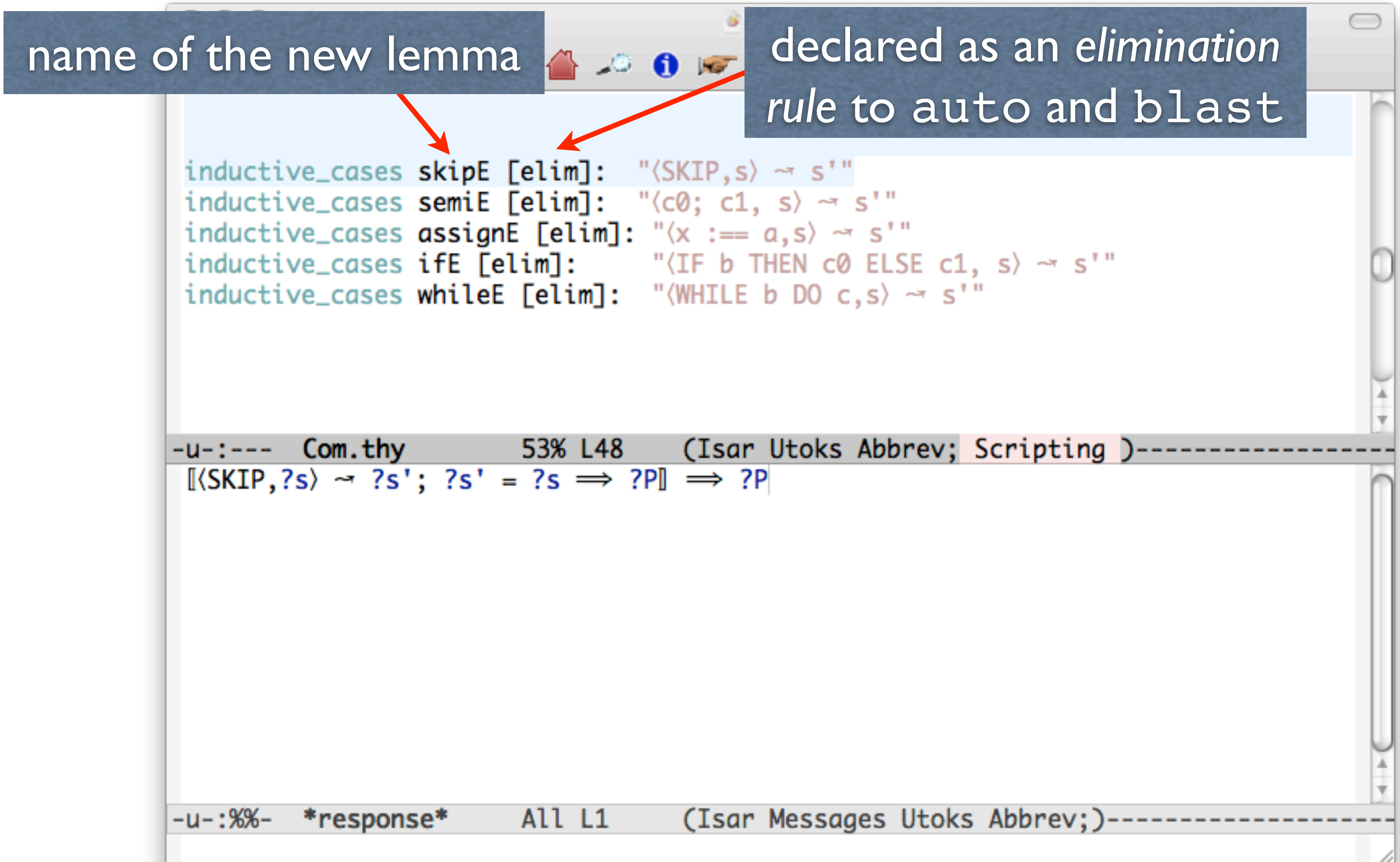

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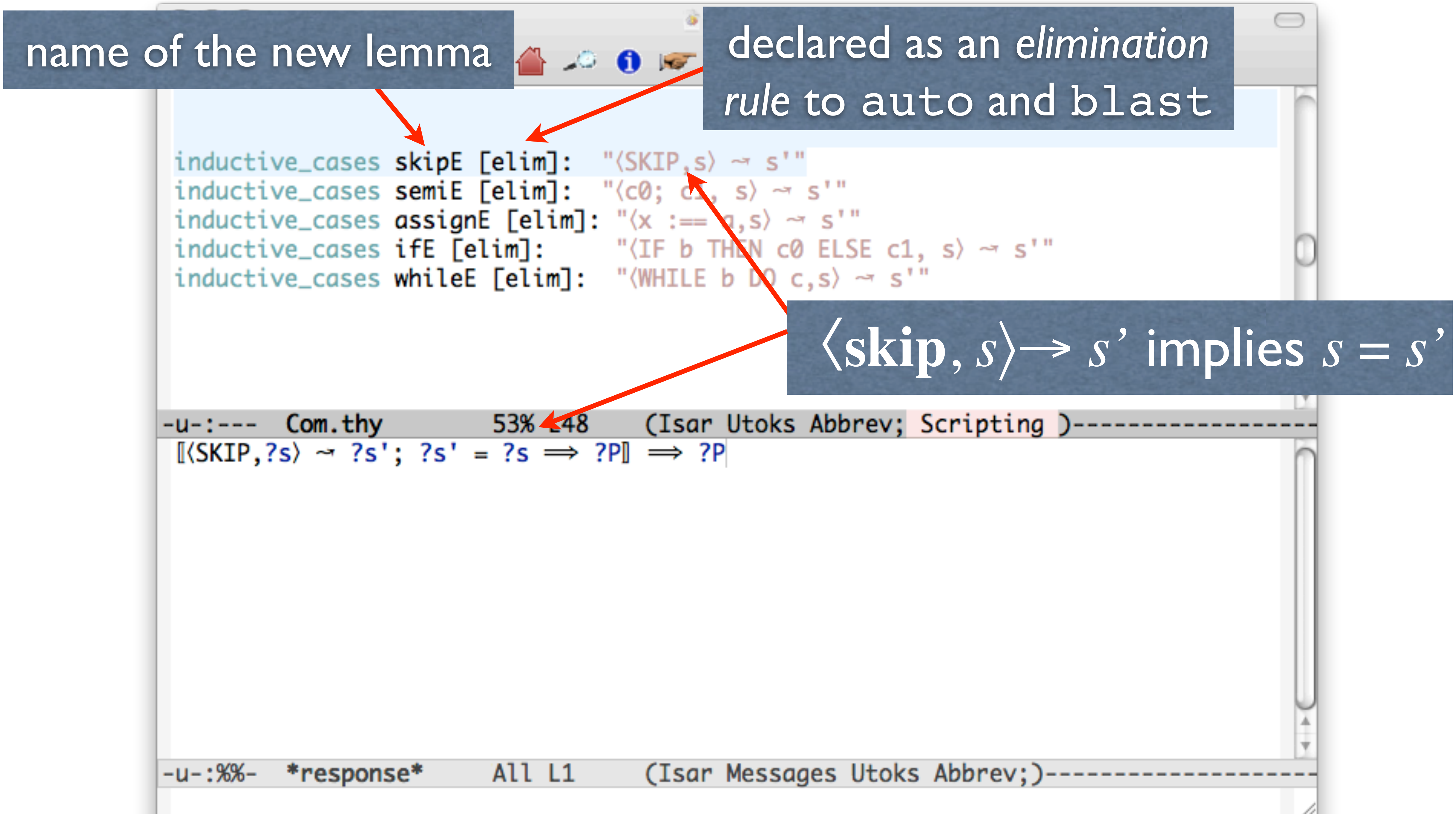

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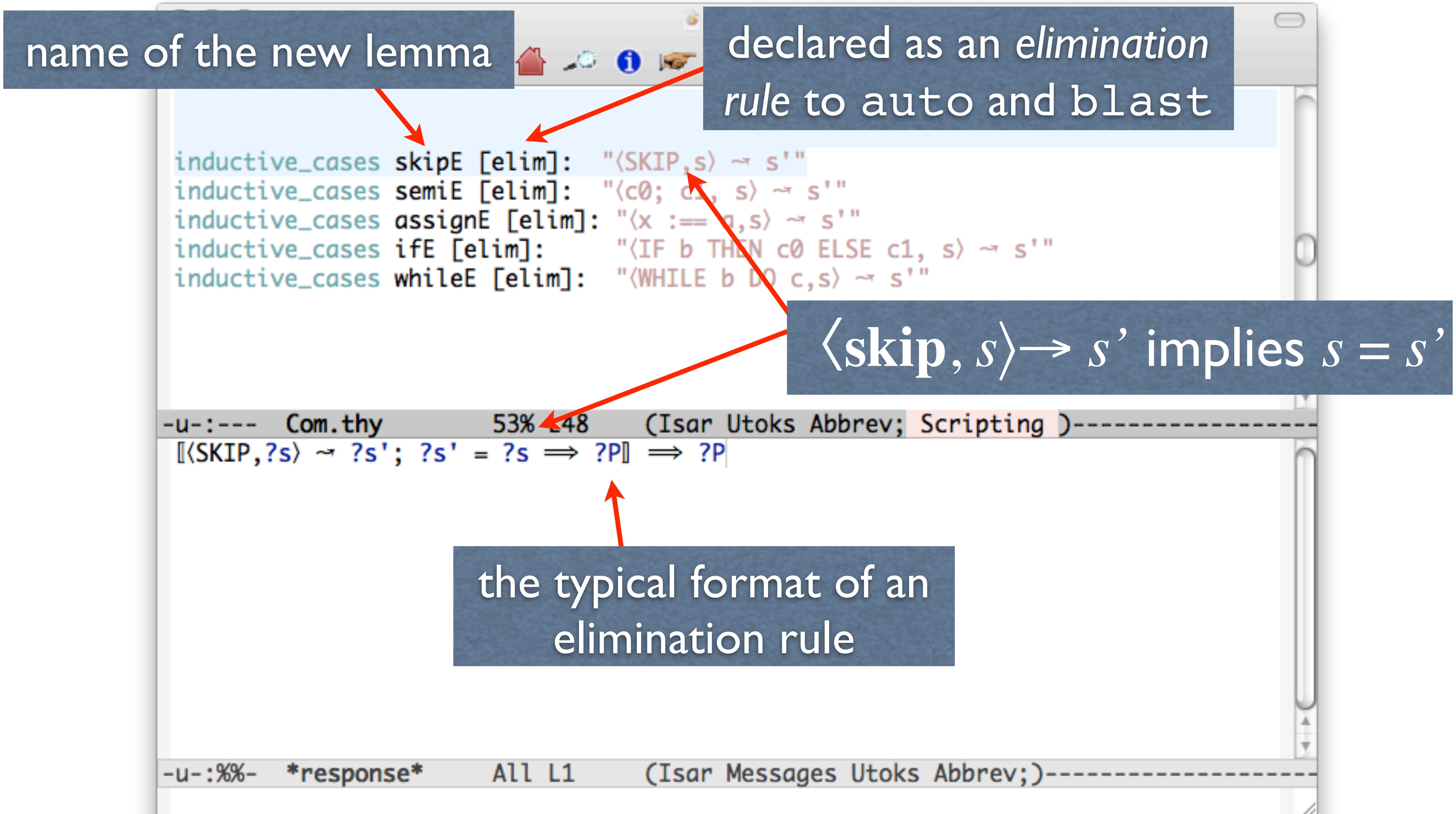
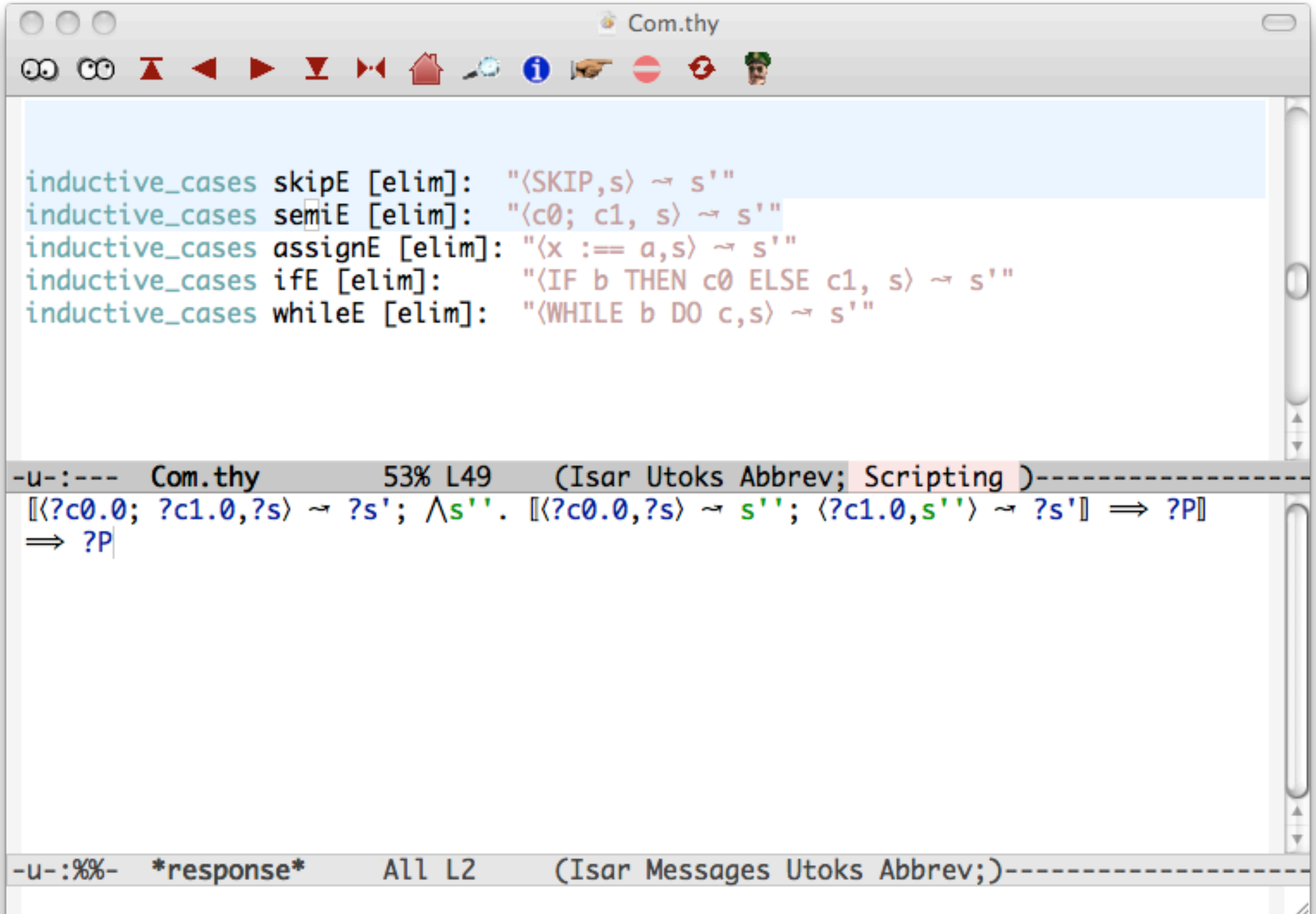
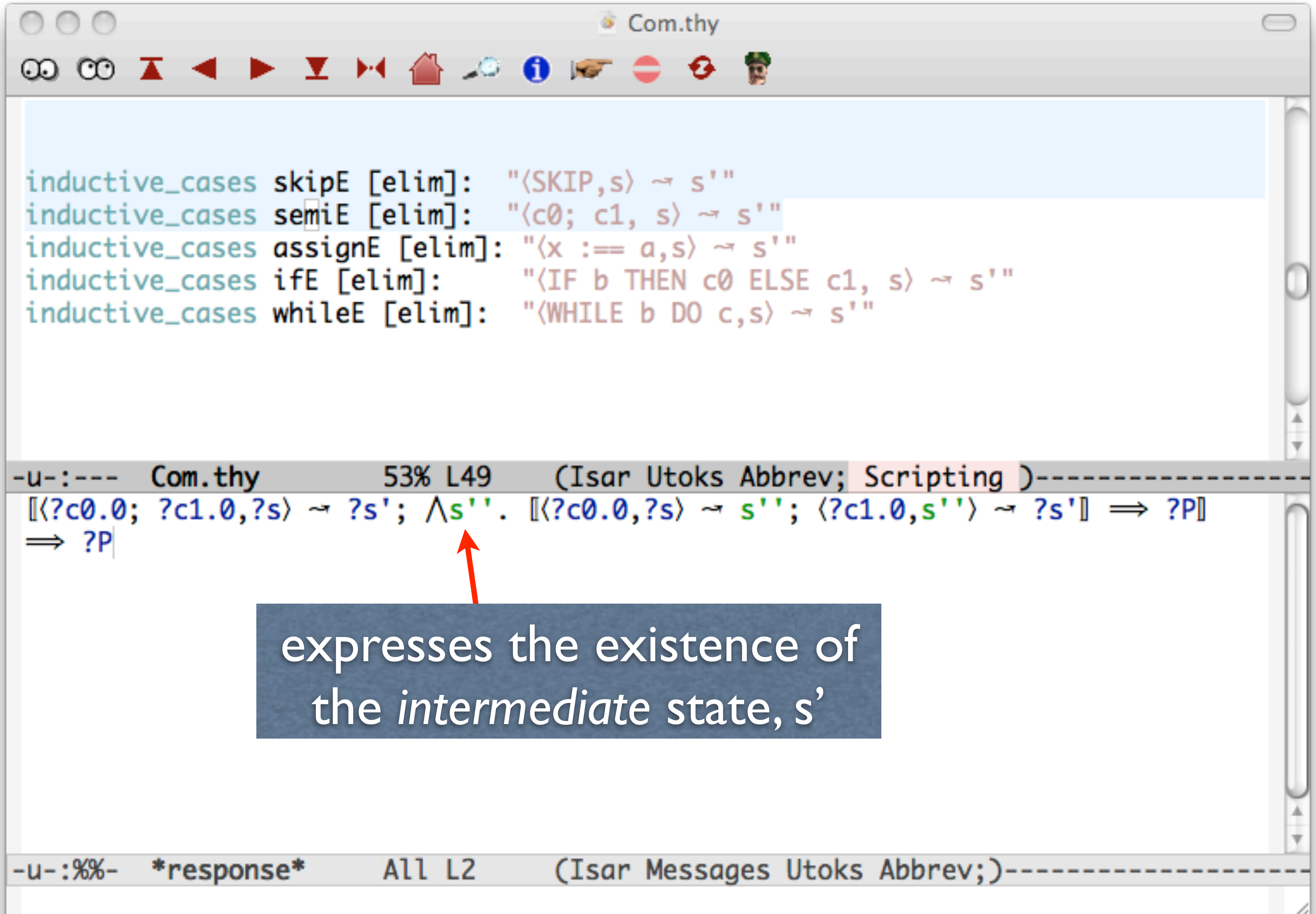


Rule Inversion Again



Rule Inversion Again



A Non-Termination Proof

$$\langle \textbf{while true do } c, s\rangle \not\to s'$$

The inductive version considers *all* possible commands

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$$\langle c, s\rangle \to s' \Longrightarrow \forall c' .\, c \neq (\textbf{while true do } c')$$

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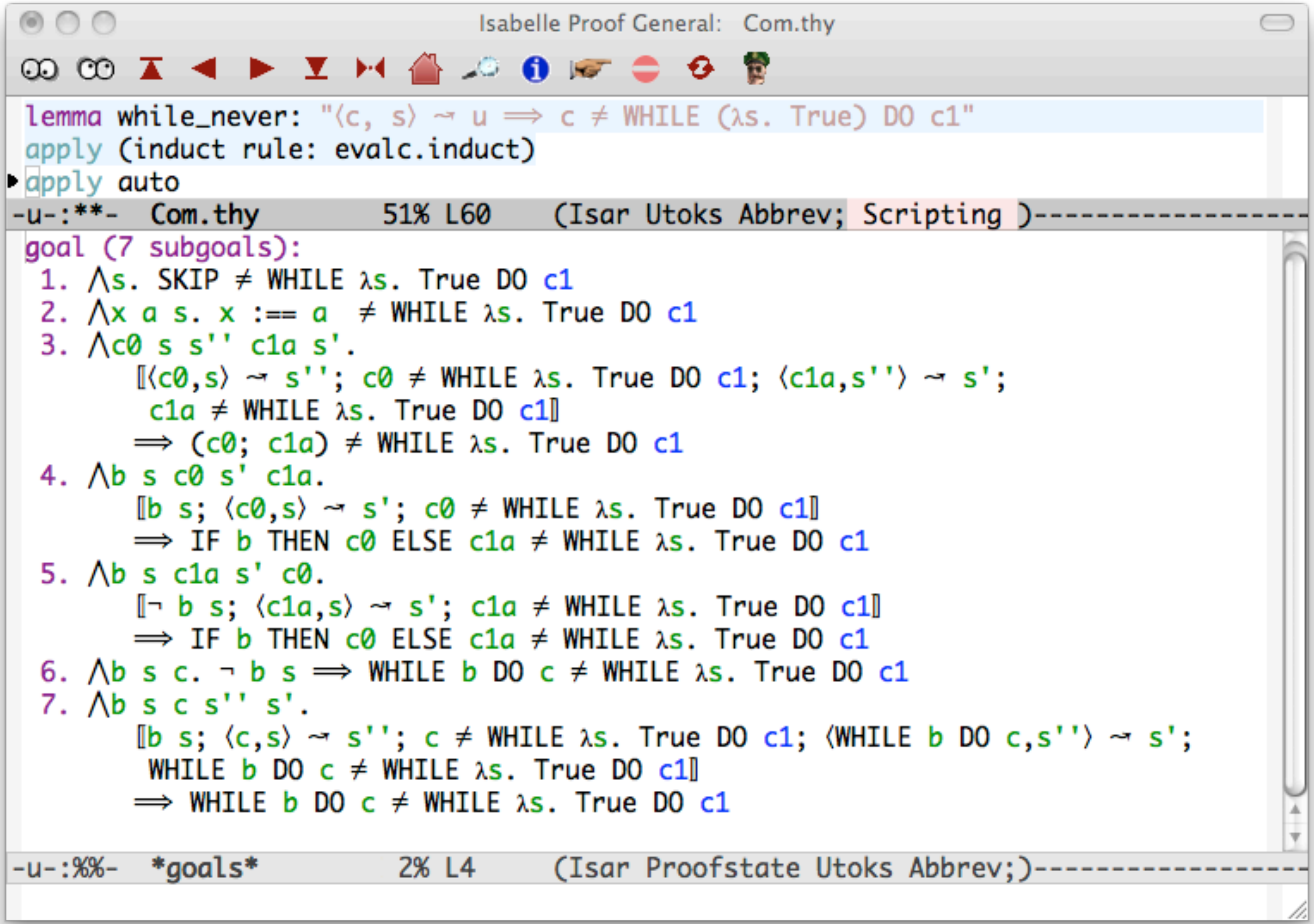
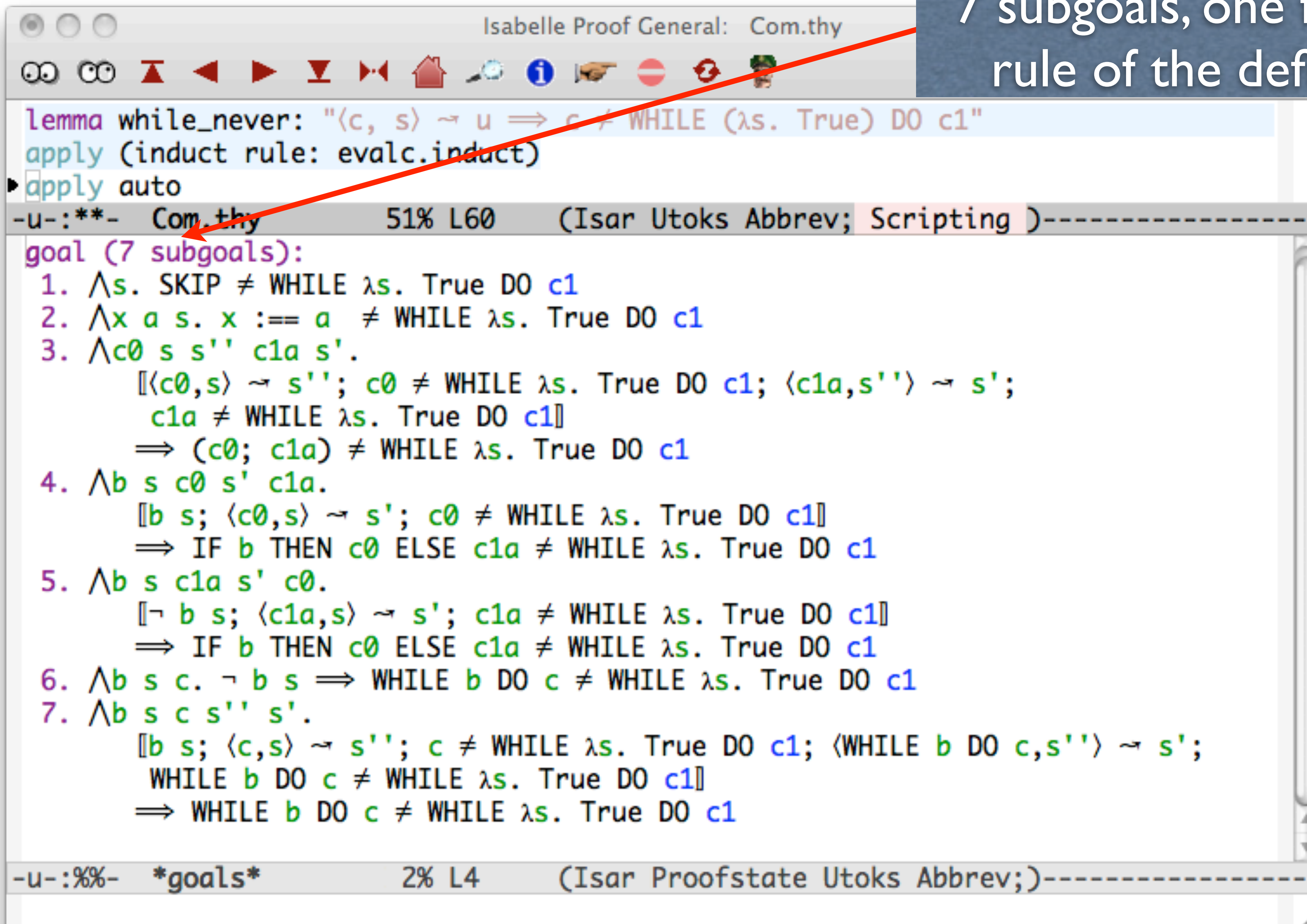
Non-Termination in Isabelle



Non-Termination in Isabelle

7 subgoals, one for each rule of the definition



Non-Termination in Isabelle

7 subgoals, one for each rule of the definition

Most are trivial, by distinctness

```
Isabelle Proof General:   Com.thy

lemma while_never: "⟨c, s⟩ ⟶ u ⟹ c ≠ WHILE (λs. True) DO c1"
apply (induct rule: evalc.induct)
apply auto
-u-:**-  Com.thy        51% L60     (Isar Utoks Abbrev; Scripting
goal (7 subgoals):
 1. ⋀s. SKIP ≠ WHILE λs. True DO c1
 2. ⋀x a s. x :== a  ≠ WHILE λs. True DO c1
 3. ⋀c0 s s'' c1a s'.
       ⟦⟨c0,s⟩ ⟶ s''; c0 ≠ WHILE λs. True DO c1; ⟨c1a,s''⟩ ⟶ s';
        c1a ≠ WHILE λs. True DO c1⟧
       ⟹ (c0; c1a) ≠ WHILE λs. True DO c1
 4. ⋀b s c0 s' c1a.
       ⟦b s; ⟨c0,s⟩ ⟶ s'; c0 ≠ WHILE λs. True DO c1⟧
       ⟹ IF b THEN c0 ELSE c1a ≠ WHILE λs. True DO c1
 5. ⋀b s c1a s' c0.
       ⟦¬ b s; ⟨c1a,s⟩ ⟶ s'; c1a ≠ WHILE λs. True DO c1⟧
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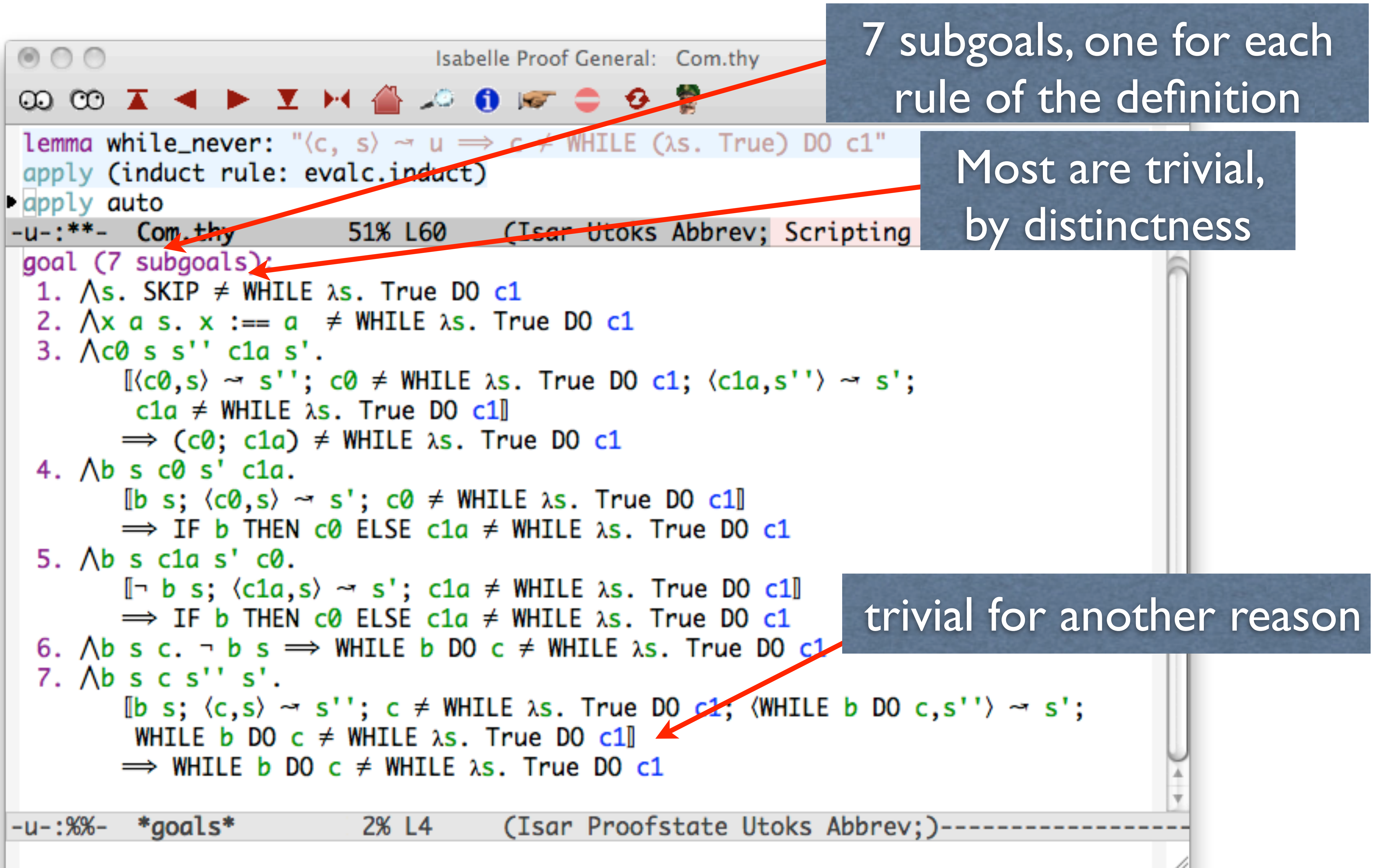
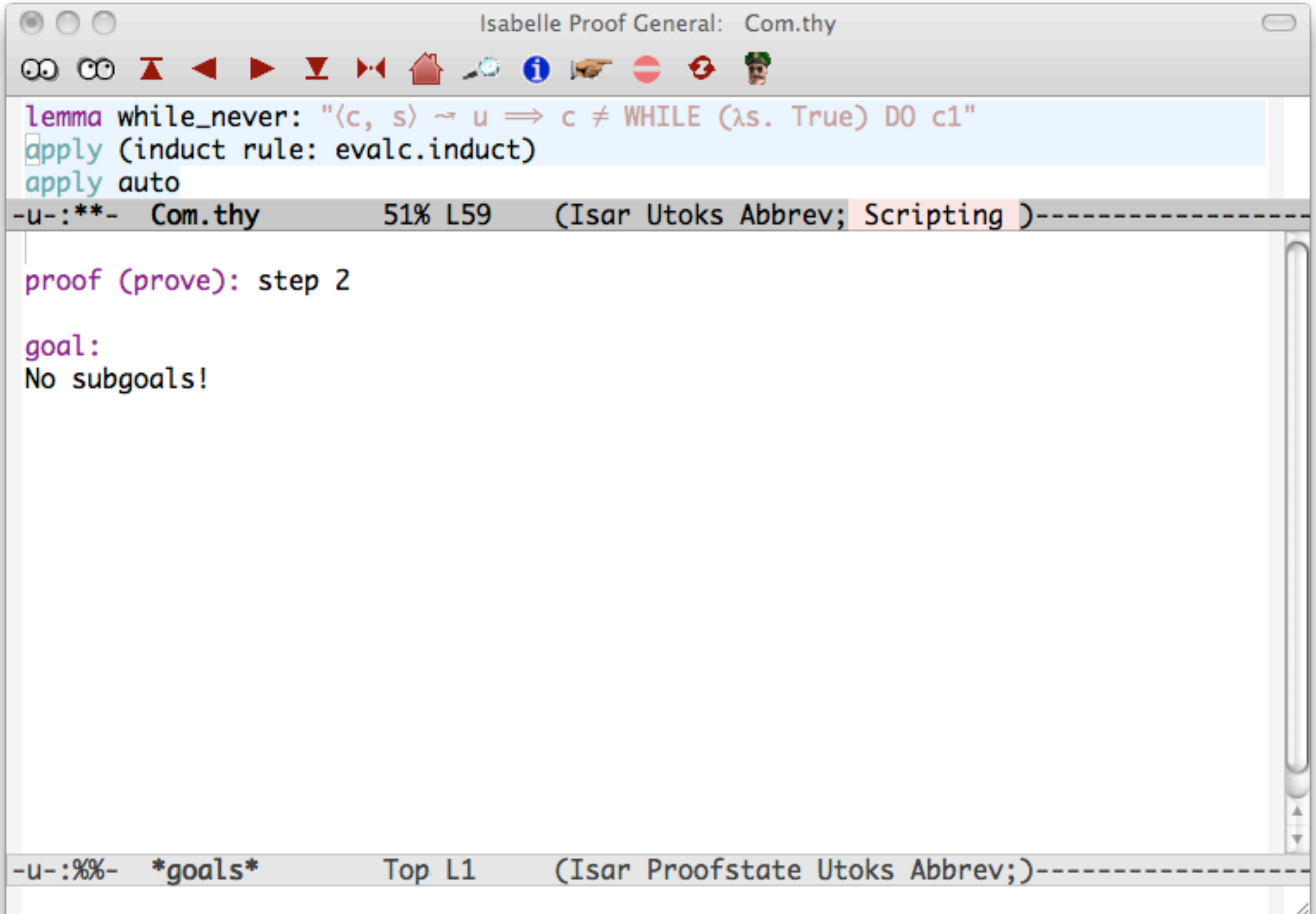
 6. ⋀b s c. ¬ b s ⟹ WHILE b DO c ≠ WHILE λs. True DO c1
 7. ⋀b s c s'' s'.
       ⟦b s; ⟨c,s⟩ ⟶ s''; c ≠ WHILE λs. True DO c1; ⟨WHILE b DO c,s''⟩ ⟶ s';
        WHILE b DO c ≠ WHILE λs. True DO c1⟧
       ⟹ WHILE b DO c ≠ WHILE λs. True DO c1

-u-:%%-  *goals*        2% L4       (Isar Proofstate Utoks Abbrev;)----------------
```

Non-Termination in Isabelle



Done!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apply (induct rule: evalc.induct)
apply auto
-u-:**- Com.thy 51% L59 (Isar Utoks Abbrev; Scripting)
proof (prove): step 2
goal:
No subgoals!
-u-:%%- *goals* Top L1 (Isar Proofstate Utoks Abbrev;)

Determinacy

$$\frac{\langle c, s\rangle \to t \qquad \langle c, s\rangle \to u}{t = u}$$

If a command is executed in a given state, and it terminates, then this final state is *unique*.

Determinacy in Isabelle

```
Com.thy
theorem com_det: "⟨c,s⟩ ⇝ t ⟹ ⟨c,s⟩ ⇝ u ⟹ u = t"
apply (induct arbitrary: u rule: evalc.induct)
apply blast+
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-u-:%%-  *goals*         3% L5      (Isar Proofstate Utoks Abbrev;)----------------
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Determinacy in Isabelle

allow the other state to *vary*

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theorem com_det: "⟨c,s⟩ ⇝ t ⟹ ⟨c,s⟩ ⇝ u ⟹ u = t"
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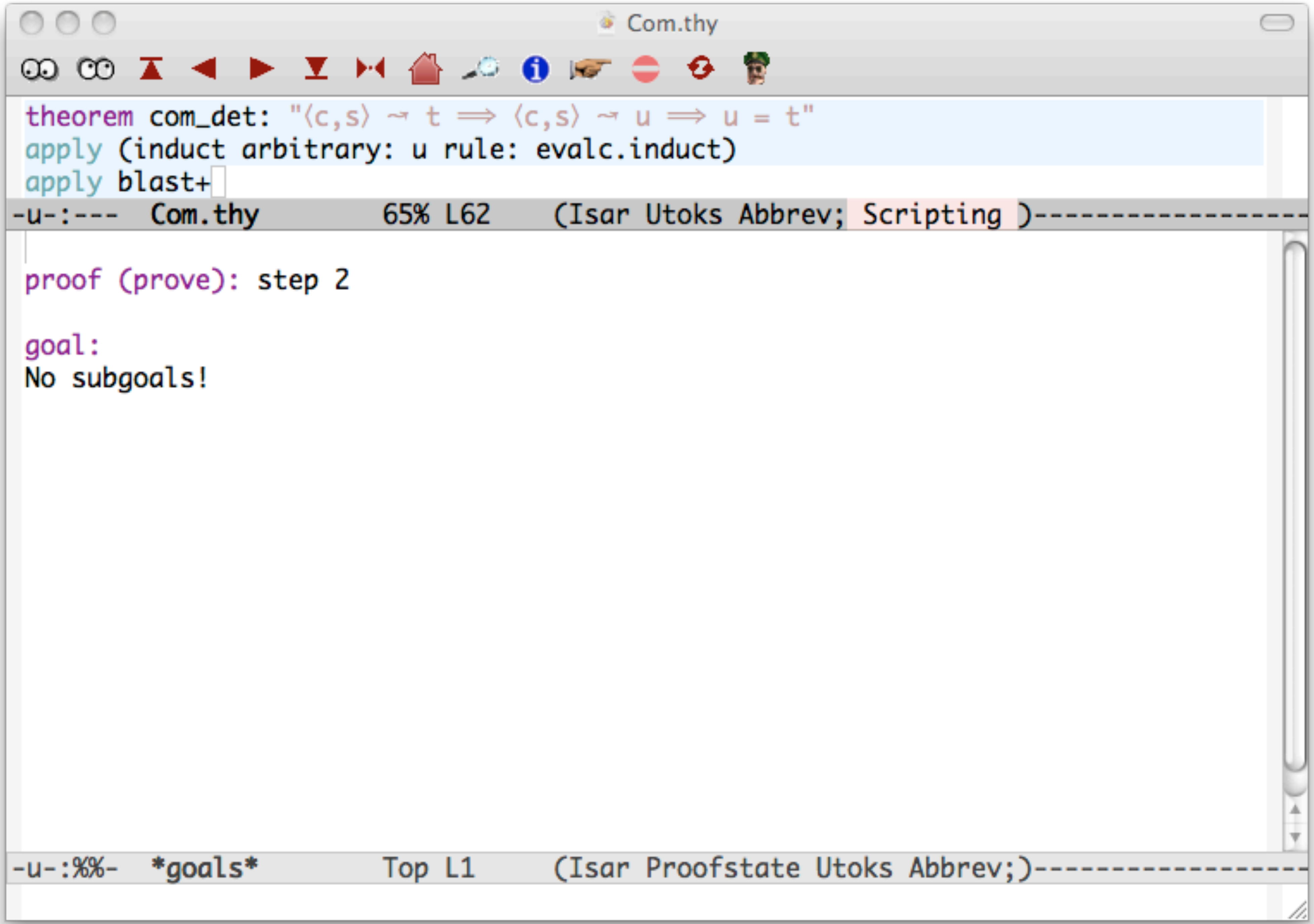
```

allow the other state to *vary*

trivial by rule inversion

Proved by Rule Inversion



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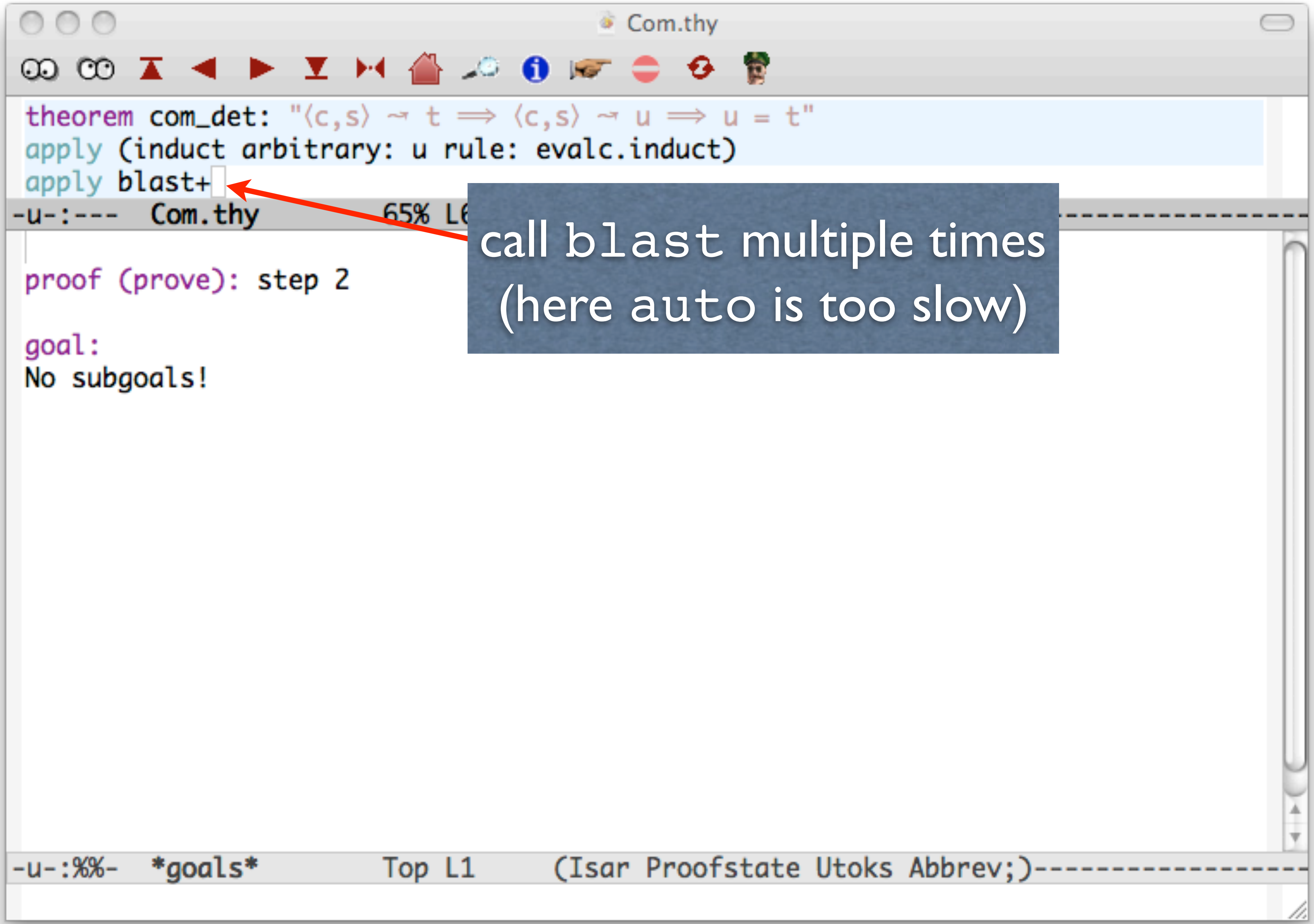

Semantic Equivalence

Isabelle Proof General: Com.thy

```
subsection {*Equivalence of commands*}

text{*Two commands are equivalent if they allow the same transitions.*}

definition
  equiv_c :: "com ⇒ com ⇒ bool" ("_ ~ _")
where
  "(c ~ c') = (∀s s'. (⟨c, s⟩ → s') = (⟨c', s⟩ → s'))"
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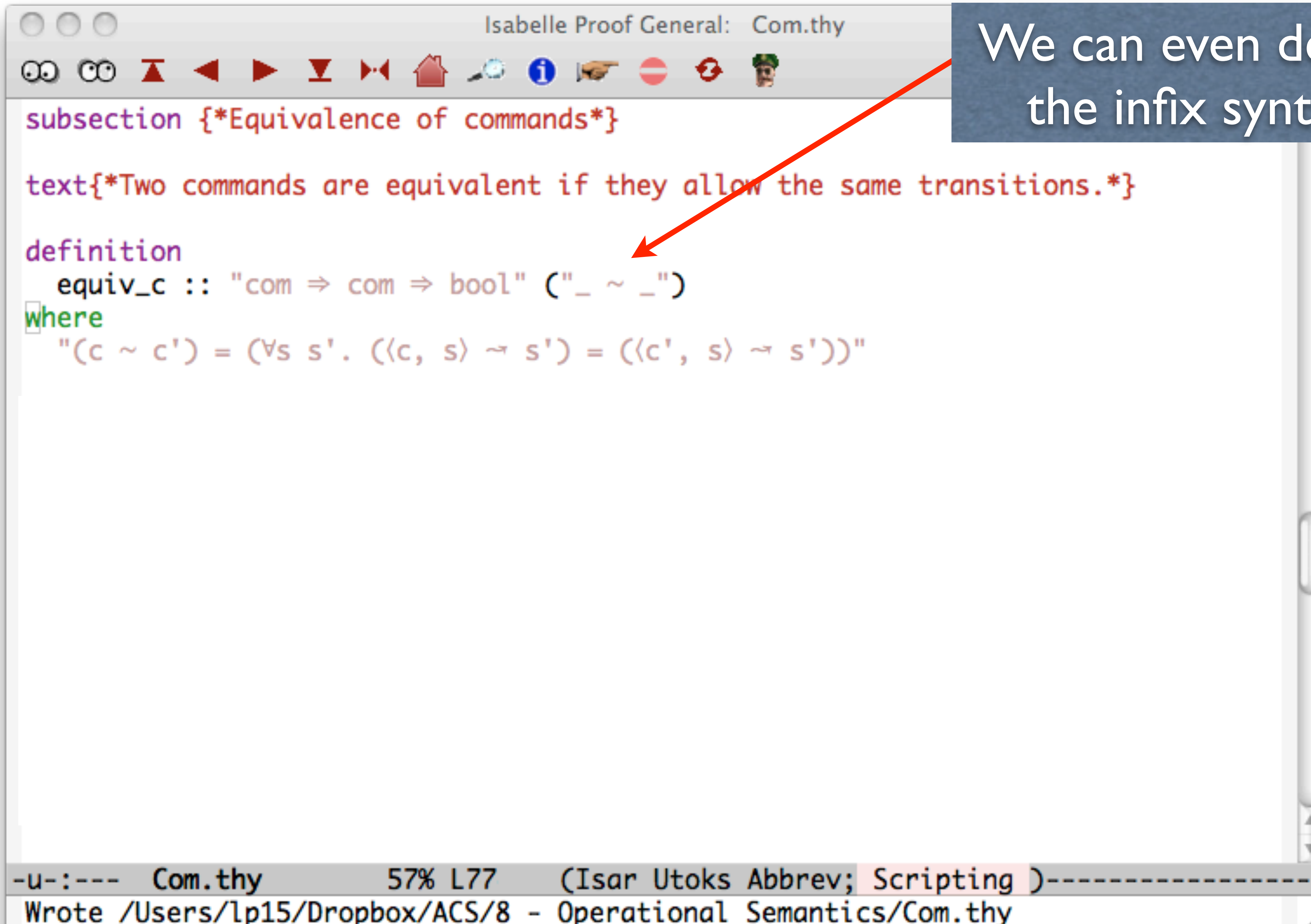

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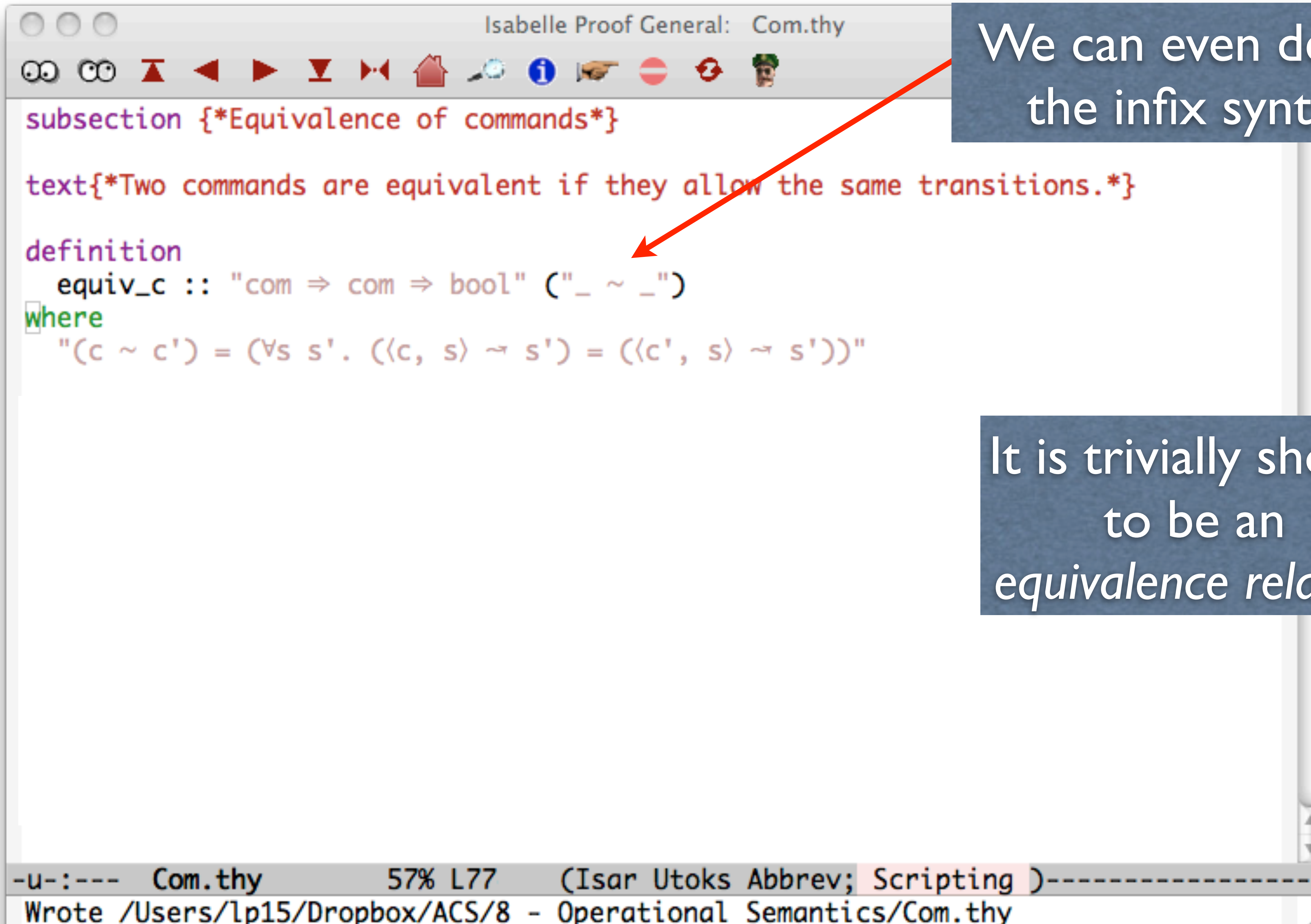

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It is trivially shown to be an *equivalence relation*

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lemma equiv_refl:
  "c ~ c"
by (auto simp add: equiv_c_def)

lemma equiv_sym:
  "c1 ~ c2 ⟹ c2 ~ c1"
by (auto simp add: equiv_c_def)

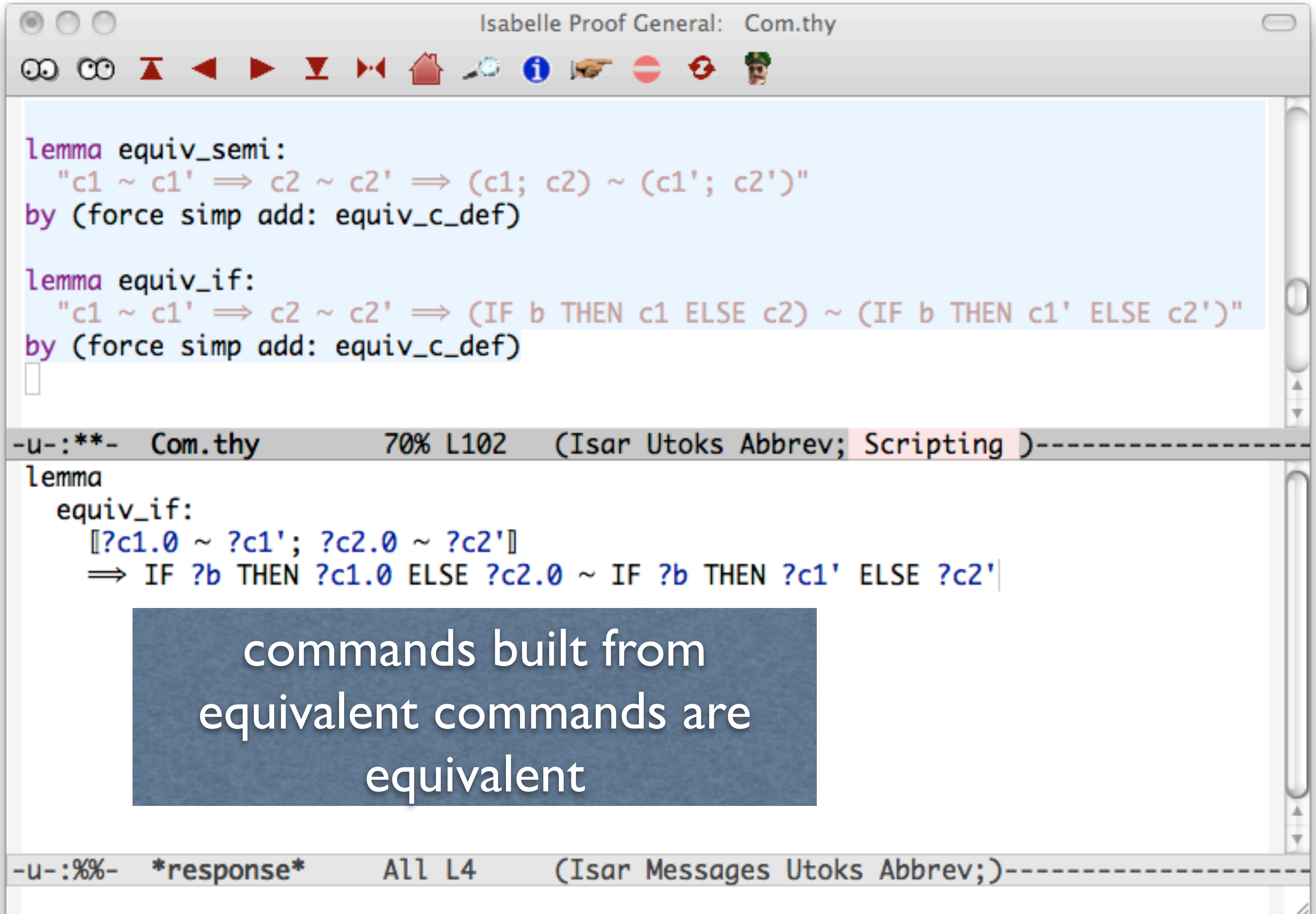
lemma equiv_trans:
  "c1 ~ c2 ⟹ c2 ~ c3 ⟹ c1 ~ c3"
by (auto simp add: equiv_c_def)

-u-:---  Com.thy        57% L77     (Isar Utoks Abbrev; Scripting )----------------
Wrote /Users/lp15/Dropbox/ACS/8 - Operational Semantics/Com.thy
```

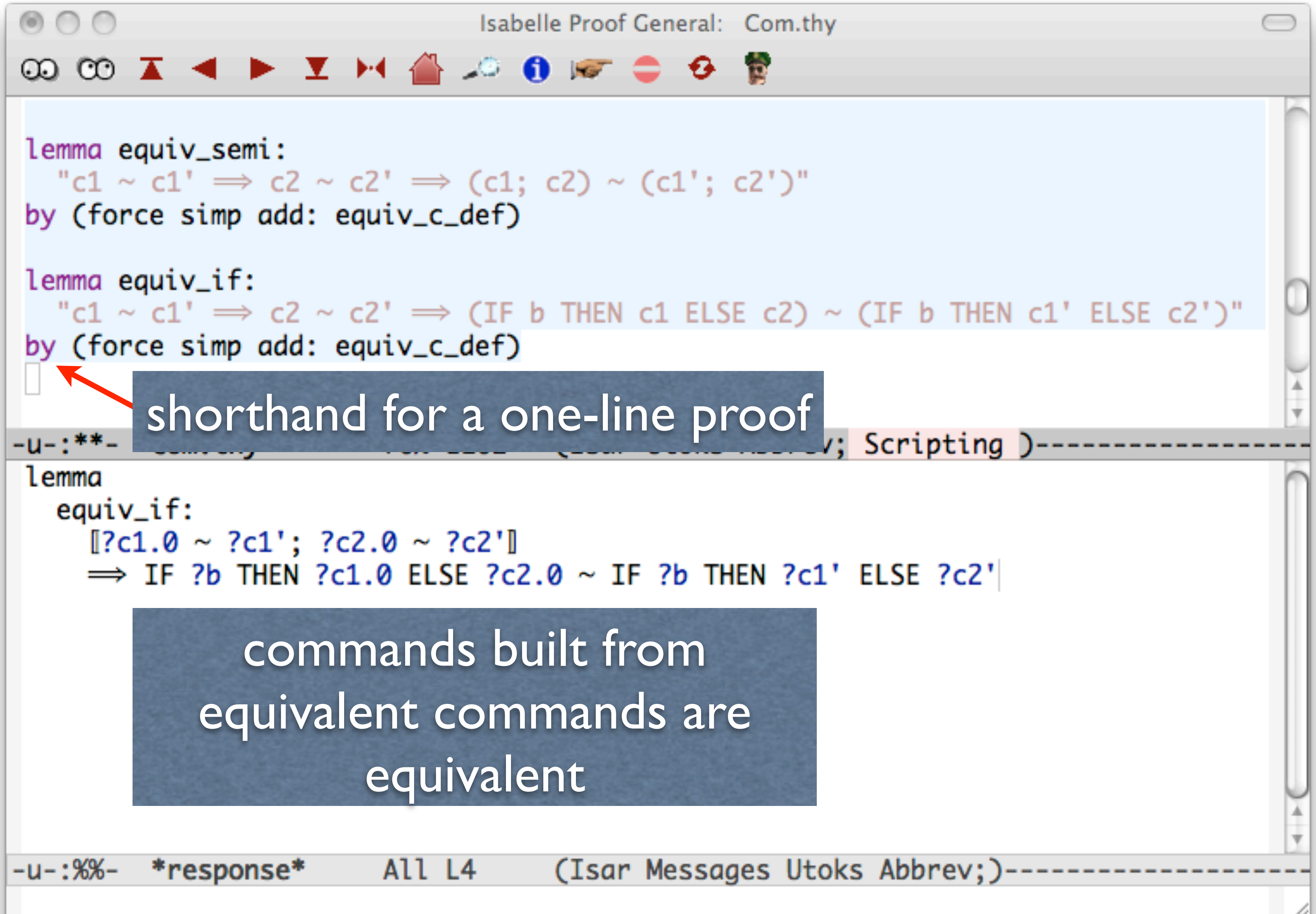
We can even define the infix syntax

It is trivially shown to be an *equivalence relation*

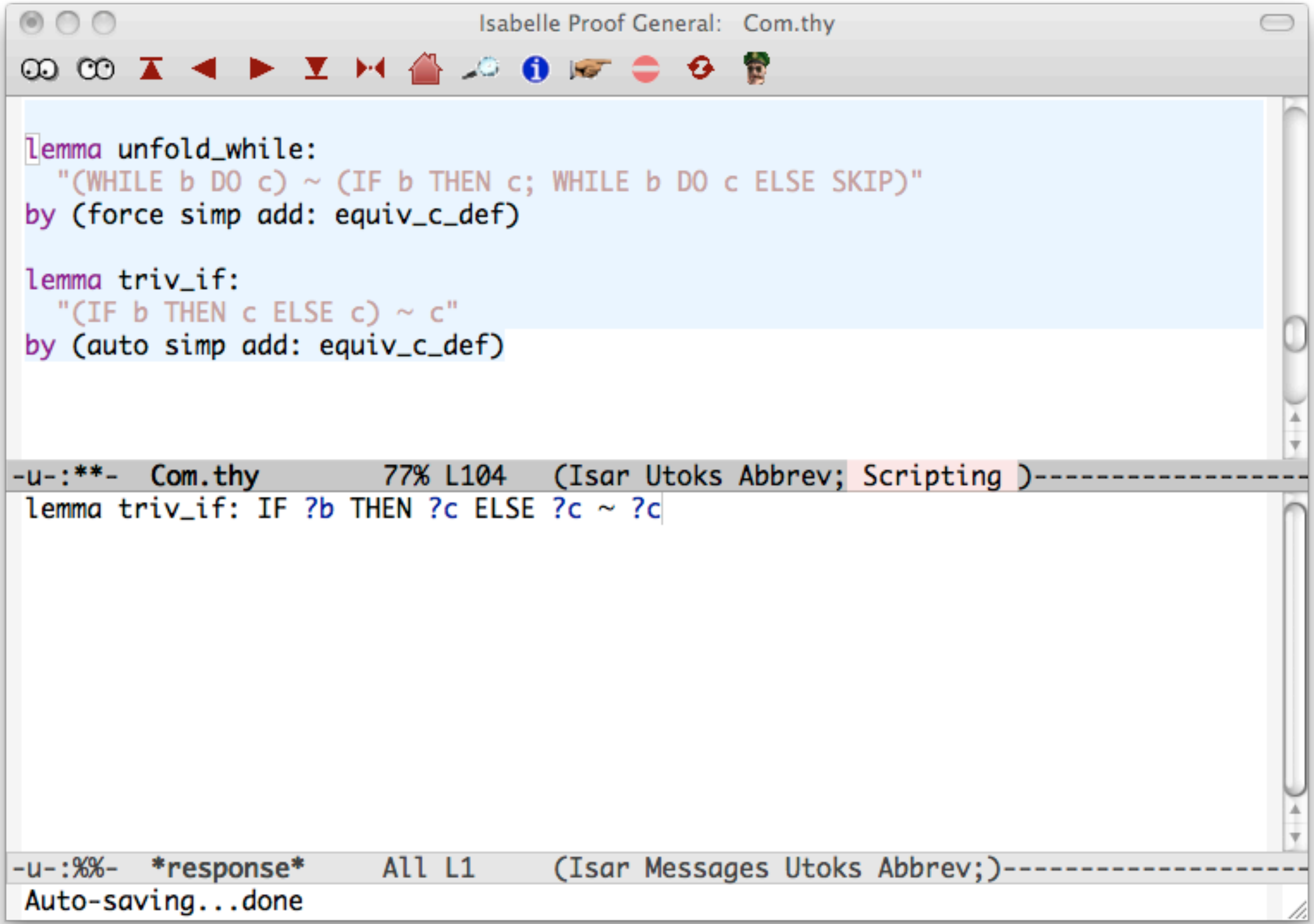
More Semantic Equivalence!



More Semantic Equivalence!



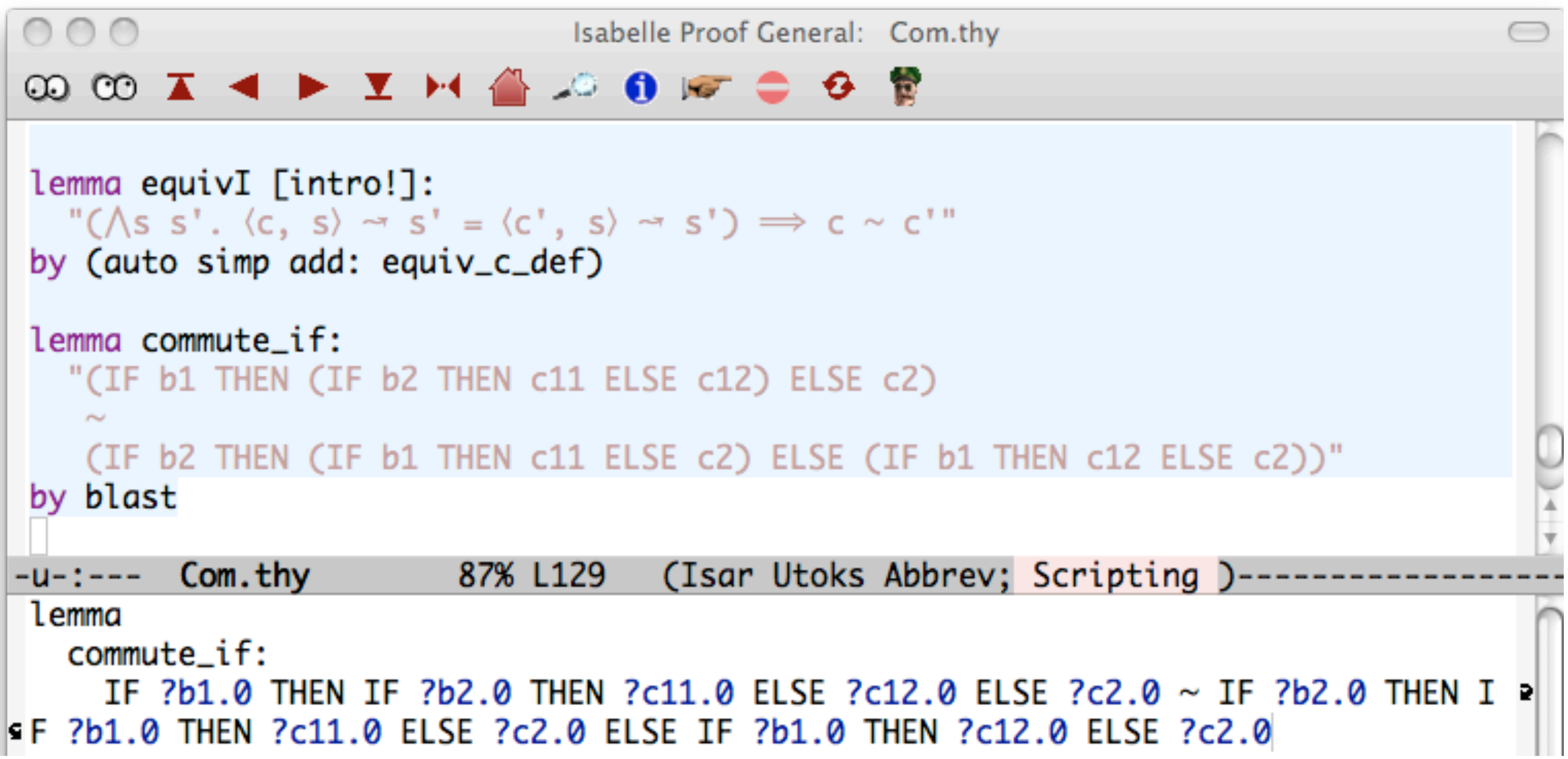
And More!!


Isabelle Proof General: Com.thy
lemma unfold_while:
"(WHILE b DO c) ~ (IF b THEN c; WHILE b DO c ELSE SKIP)"
by (force simp add: equiv_c_def)
lemma triv_if:
"(IF b THEN c ELSE c) ~ c"
by (auto simp add: equiv_c_def)
-u-:**- Com.thy 77% L104 (Isar Utoks Abbrev; Scripting)
lemma triv_if: IF ?b THEN ?c ELSE ?c ~ ?c
-u-:%%- *response* All L1 (Isar Messages Utoks Abbrev;)
Auto-saving...done

A New Introduction Rule

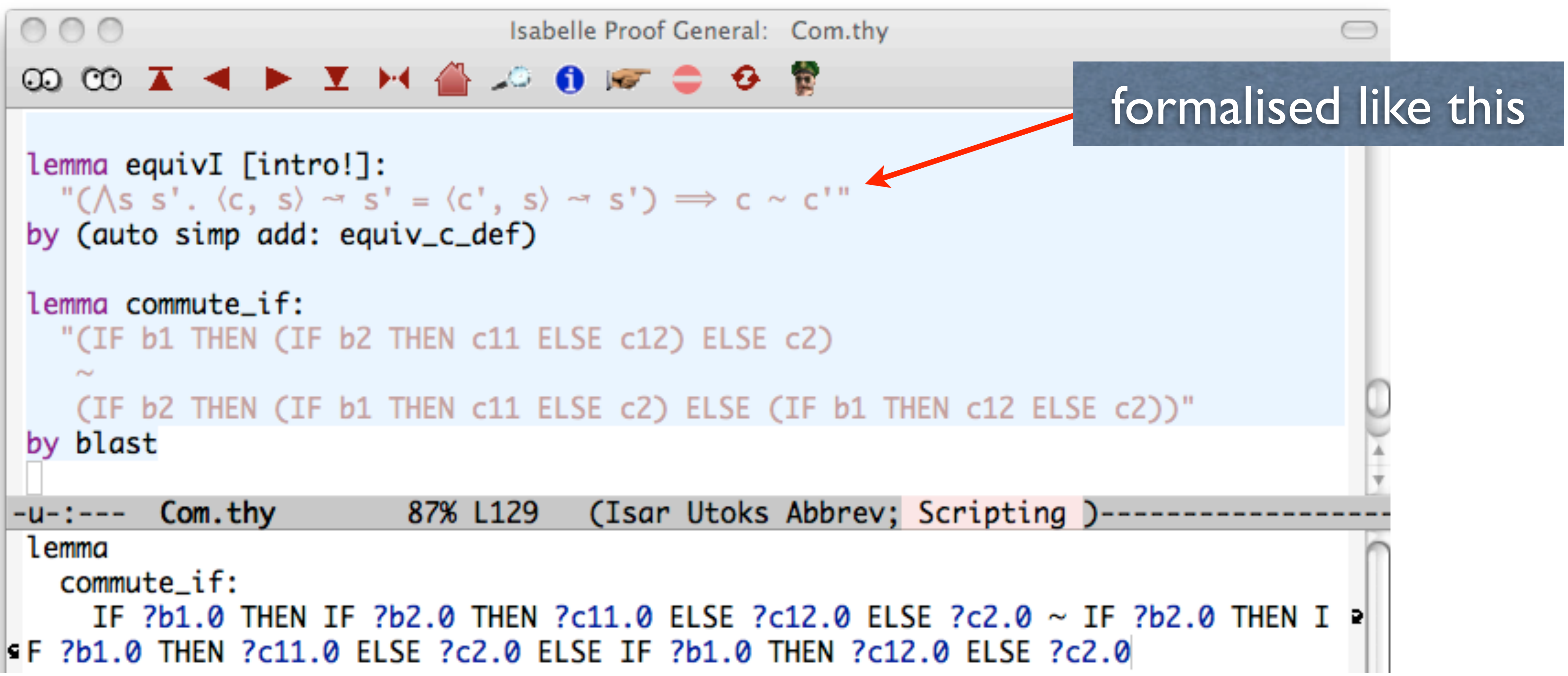
$$\frac{\langle c, s\rangle \to s' \iff \langle c', s\rangle \to s'}{c \sim c'}$$

s and s' not free…



A New Introduction Rule

$$\frac{\langle c, s\rangle \rightarrow s' \iff \langle c', s\rangle \rightarrow s'}{c \sim c'} \qquad s \text{ and } s' \text{ not free}\ldots$$



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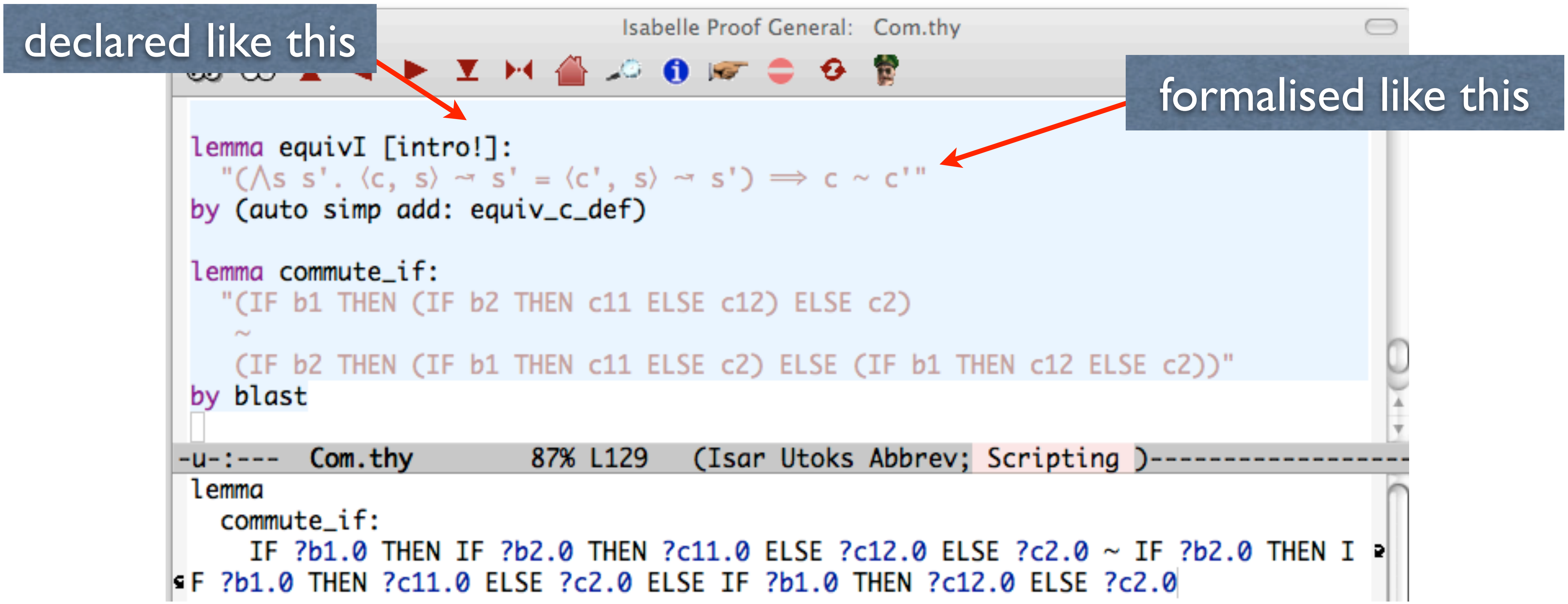
$$\frac{\langle c, s\rangle \rightarrow s' \iff \langle c', s\rangle \rightarrow s'}{c \sim c'} \qquad s \text{ and } s' \text{ not free...}$$



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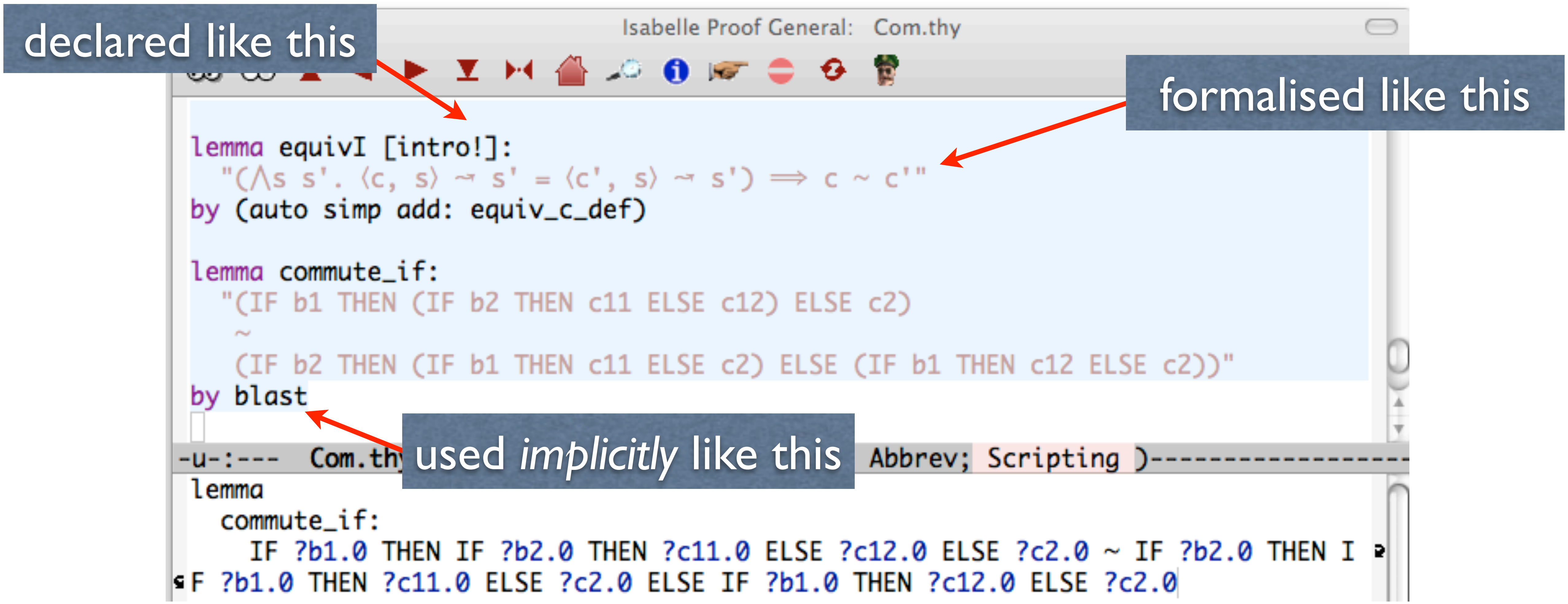
$$\frac{\langle c, s\rangle \to s' \iff \langle c', s\rangle \to s'}{c \sim c'} \quad s \text{ and } s' \text{ not free...}$$



Final Remarks on Semantics

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- Variable binding is crucial in larger examples, and should be formalised using the *nominal package*.
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    - renaming bound variables consistently
- Serious proofs will be complex and difficult!